Orbital Parameters and Distances

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1 Introduction

Using pulsar scintillometry to resolve inclinations and distances would be helpful for pulsar timing arrays (since precise distances allow the pulsar term to be used) as well as questions of the neutron star equation of state (see Marten's paper on B1957's mass for how the inclination can help secure mass measurements). One way to access this information, for binary pulsars, is by looking at the variation in arc curvature due to both the Earth and Pulsar orbits.

2 Circular Orbits

2.1 A Single Screen

For the simplest case, we consider a pulsar in a circular binary assuming a circular Earth orbit.

We begin by expressing the pulsar's orbital velocity in its orbital plane as

$$\begin{aligned} v_x &= -\frac{a_p 2\pi}{P_b}\sin(\theta_p) \\ v_y &= \frac{a_p 2\pi}{P_b}\cos(\theta_p) \end{aligned} \tag{1}$$

where θ_p is the pulsar orbital phase starting at the ascending node, which is known from timing. However, a_p is generally unknown and instead we measure the projected semi-major axis $x = a_p \sin(i_p)$ or the radial-velocity amplitude

$$K_p = \frac{2\pi x}{P_b} \tag{2}$$

Substituting in K_p and projecting onto the sky with the x2 axis along the line of nodes gives

$$\begin{aligned} v_{x2} &= -\frac{K_p}{\sin(i_p)}\sin(\theta_p) \\ v_{y2} &= \frac{K_P}{\sin(i_p)}\cos(\theta_p)\cos(i_p) \end{aligned} \tag{3}$$

Parameter	Symbol	Known
Semi-major Axis of Binary	a_p	✗
Projected Semi-major Axis of Binary	x	✓
Pulsar Inclination	i_p	✗
Binary Period	P_b	✓
Pulsar Radial Velocity Amplitude	K_p	✓
Pulsar Distance	d_p	✗
Pulsar Orbital Orientation	Ω_p	✗
Pulsar Proper Motion (Right Ascension)	μ_α	✓
Pulsar Proper Motion (Declination)	μ_δ	✓
Screen Distance	d_s	✗
Screen Orientation	Ω_s	✗
Screen Velocity	v_{ISM}	✗
Earth Orbital Speed	$\bar{v}_e$	✓
Earth Orbital Inclination	i_e	✓
Earth Orbital Orientation	Ω_e	✓
Observing Wavelength	λ	✓
Arc Curvature	η	✓

Table 1: A list of physical parameters used in the simple model

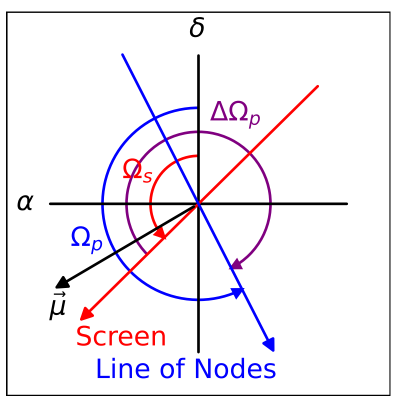


Figure 1: Schematic of angles on the sky. Arrows on the Line of Nodes and Screen represent the positive direction along those axes. For the Line of Nodes this is towards the ascending node, and is arbitrarily chosen for the screen.

Next, we would like to rotate into right ascension and declination

$$\begin{aligned} v_{p\alpha} &= -\frac{K_p}{\sin(i_p)}(\sin(\Omega_p)\sin(\theta_p) - \cos(i_p)\cos(\Omega_p)\cos(\theta_p)) \\ v_{p\delta} &= -\frac{K_p}{\sin(i_p)}(\cos(i_p)\sin(\Omega_p)\cos(\theta_p) + \cos(\Omega_p)\sin(\theta_p)) \end{aligned} \tag{4}$$

and finally project onto the screen

$$\begin{aligned} v_p = -\frac{K_p}{\sin(i_p)}[&\cos(\theta_p)\cos(i_p)(\cos(\Omega_s)\sin(\Omega_p) - \sin(\Omega_s)\cos(\Omega_p)) \\ &+ \sin(\theta_p)(\cos(\Omega_s)\cos(\Omega_p) + \sin(\Omega_s)\sin(\Omega_p)] \end{aligned} \tag{5}$$

which simplifies to

$$v_p = \frac{K_p}{\sin(i_p)}(\cos(i_p)\sin(\Delta\Omega_p)\cos(\theta_p) - \cos(\Delta\Omega_p)\sin(\theta_p)) \tag{6}$$

where $\Delta\Omega_p = \Omega_s - \Omega_p$. Similarly for Earth,

$$v_e = \bar{v}_e(\cos(i_e)\sin(\Delta\Omega_e)\cos(\theta_e) - \cos(\Delta\Omega_e)\sin(\theta_e)) \tag{7}$$

The effective velocity is then given by

$$v_{eff} = \frac{1}{s}v_{ISM} - (v_p + \mu_\alpha d_p \cos(\Omega_s) + \mu_\delta d_p \sin(\Omega_s))\frac{d_{eff}}{d_p} - v_e \tag{8}$$

where the fractional and effective distances are given by

$$\begin{aligned} s &= 1 - \frac{ds}{dp} \\ d_{eff} &= \frac{d_p d_s}{d_p - d_s} \end{aligned} \tag{9}$$

Since curvature is given by

$$\eta = \frac{\lambda^2 d_{eff}}{2cv_{eff}^2} \tag{10}$$

we can write

$$\begin{aligned} \frac{\lambda}{\sqrt{2\eta c}} &= \pm\frac{v_{eff}}{\sqrt{d_{eff}}} \\ &= A_{sp}\sin(\theta_p) + A_{cp}\cos(\theta_p) + A_{se}\sin(\theta_e) + A_{ce}\cos(\theta_e) + v_c \end{aligned} \tag{11}$$

where

$$
\begin{aligned}
A_{sp} &= \frac{K_p \cos(\Delta\Omega_p)\sqrt{d_{eff}}}{d_p \sin(i_p)} \\
A_{cp} &= -\frac{K_p \sin(\Delta\Omega_p)\sqrt{d_{eff}}}{d_p \tan(i_p)} \\
A_{se} &= \frac{\bar{v}_e \cos(\Delta\Omega_e)}{\sqrt{d_{eff}}} \\
A_{ce} &= -\frac{\bar{v}_e \sin(\Delta\Omega_e)\cos(i_e)}{\sqrt{d_{eff}}} \\
v_c &= \frac{V_{ISM}}{s\sqrt{d_{eff}}} - \sqrt{d_{eff}}(\mu_\alpha \cos(\Omega_s) + \mu_\delta \sin(\Omega_s))
\end{aligned}
\tag{12}
$$

Since we only measure v_{eff}^2, we are left with 2 solutions that correspond to rotating the screen by 180°. Either solution is acceptable, so we will use the positive option.

2.1.1 Extracting Physical Parameters

The first physical parameter we can extract is Ω_s.

$$\tan(\Delta\Omega_e) = -\frac{A_{ce}}{A_{se}\cos(i_e)} \tag{13}$$

gives two possible solutions for $\Delta\Omega_e$ and the sign of $A_s e$ fixes which one. Using the known value of Ω_e then gives Ω_s. The value of d_{eff} can be extracted from A_{se}. The remaining physical parameters i_p, Ω_p, d_p (or d_s or s), and v_{ISM} are degenerate, but we can extract several useful relationships between them. Firstly,

$$\tan(\Delta\Omega_p) = -\frac{A_{cp}}{A_{sp}\cos(i_p)} \tag{14}$$

gives two potential curves in i_p-Ω_p space (since we know Ω_s) with only one giving the correct sign for A_{sp}. Next, we can use this relationship to find the d_p-i_p relationship

$$d_p = \frac{K_p \cos(\Delta\Omega_p)\sqrt{d_{eff}}}{A_{sp}\sin(i_p)} \tag{15}$$

Finally, v_{ISM} is related to s by

$$v_{ISM} = s[v_c\sqrt{d_{eff}} + d_{eff}(\mu_\alpha \cos(\Omega_s) + \mu_\delta \sin(\Omega_s))] \tag{16}$$

Knowing either angle will uniquely solve for all parameters. However, for determining the distance one need only know $\sin(i_p)$; and if one knows the distance, one can find $\sin(i)$ and $|\Delta\Omega_p|$. If one knows $\sin(i_p)$, then there are two possible solutions for i_p that are symmetric about 90°. From Eq. 14, it follows that we know the amplitude but not the sign of $\tan(\Delta\Omega_p)$. Thus, there are two solutions

Parameter	Value
d_p	0.15679kpc
i_p	137.56°
Ω_p	207°
x	3.3667144 lts
T_0	54501.4671
P_b	5.7410459 day
d_s	.0906kpc
Ω_s	134.6°
V_{ISM}	-31.9 km s^{-1}
μ_α	121.4385 mas year^{-1}
μ_δ	-71.4754 mas year^{-1}

Table 2: Physical Parameters for the Single Screen Example

for $\Delta\Omega_p$ (discounting those that give the wrong sign for A_{sp}) symmetric about 0°. This corresponds to mirroring the orbit about the screen. Both solutions will have the same value of $\cos(\Delta\Omega_p)$, and so we can use Eq. 15 to solve for dp.

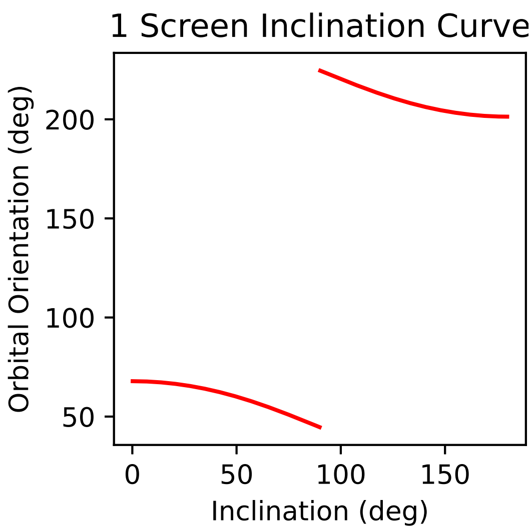
2.1.2 Example with J0437-4715

To show how this works in practice, we consider a single screen between us and PSR J0437-4715 (the pulsar used by Daniel Reardon) using the parameters in Table 2. The four A_x parameters and v_c were all recovered in a fit, but with the opposite sign. As mentioned above, this is expected behaviour and so we proceed using the fitted values. d_{eff} and Ω_s were both extracted from the Earth Orbital Coefficients. Fig. 2a shows relation between i_p and Ω_p when no distance is known. The orientation is restricted, but we cannot solve for either parameter. In Fig. 2b, we see the relationship between the distance and $\sin(i_p)$. As the pulsar becomes more face on, the distance grows. In Fig. 2c we can see the two potential solutions that arise for the inclination and orientation even when the distance is know. However, Fig 2d shows how both solutions correspond to the same $\sin(i)$ with orientations mirrored about the screen.

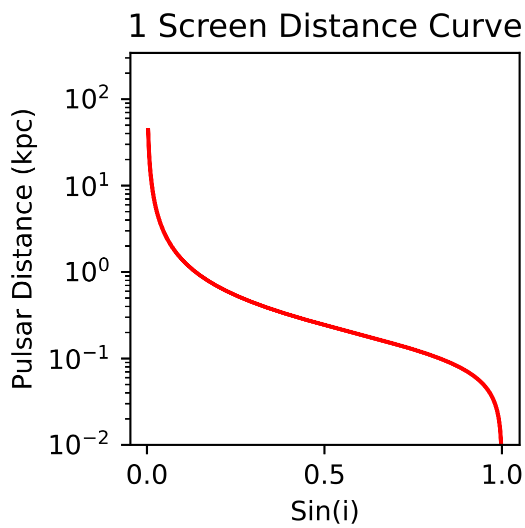
2.2 Two (or more) screens

In cases where nothing is known of the inclination or distance, parameter recovery is still possible if multiple screens can be observed. Using $\sim$ to denote parameters of the second screen,

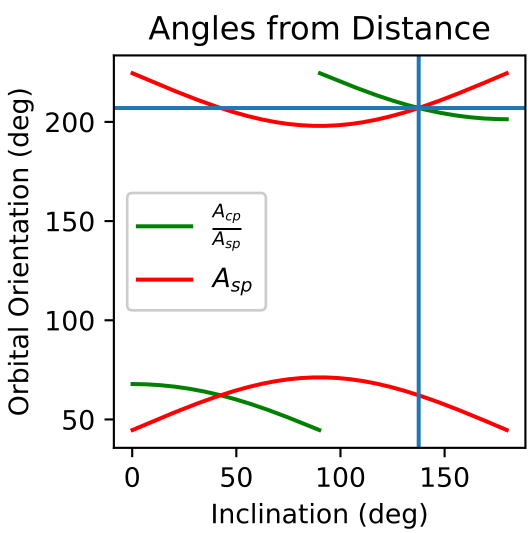
$$\frac{A_{sp}\sqrt{\widetilde{d}_{eff}}}{\widetilde{A}_{sp}\sqrt{d_{eff}}} = \frac{\cos(\Delta\Omega_o)}{\cos(\widetilde{\Delta\Omega_o})} \tag{17}$$



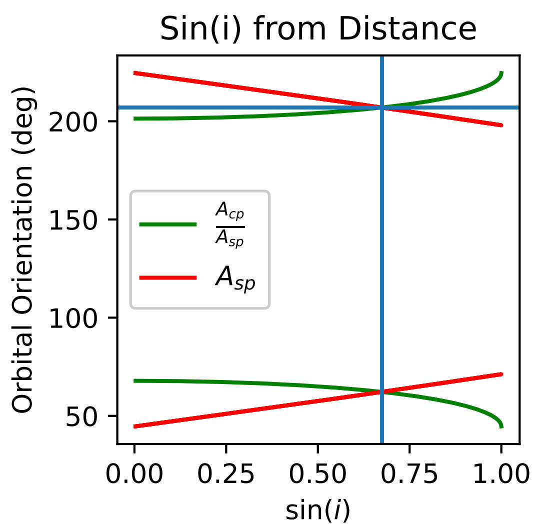
(a) Relationship between i_p and Ω_p



(b) Relationship between $\sin(i_p)$ and d_p



(c) Relationship between i_p and Ω_p including A_sp curve from a known distance.



(d) Relationship between $\sin(i_p)$ and Ω_p. Note that there are two solutions, but they both share the same value of $\sin(i)$

Figure 2: Parameter relationships for a single screen.

which we can rewrite as

$$\tan(\Omega_o) = \frac{\frac{A_{sp}\sqrt{\widetilde{d}_{eff}}}{\widetilde{A}_{sp}\sqrt{d_{eff}}}\cos(\widetilde{\Omega}_s) - \cos(\Omega_s)}{\sin(\Omega_s) - \frac{A_{sp}\sqrt{\widetilde{d}_{eff}}}{\widetilde{A}_{sp}\sqrt{d_{eff}}}\sin(\widetilde{\Omega}_s)} \tag{18}$$

Giving two solutions for Ω_o offset by 180°. We will use that the inclination is between 0 and 180 degrees (the orbit is unchanged by the transformation $i_p \Rightarrow -i_p$ and $\Omega_o \Rightarrow \Omega_o + 180°$). Since $\sin(i_p)$ is non negative, it follows that only one of these solutions will yield the right sign for A_{sp}. Since $\cos(i_p)$ is monotonic this will give a unique inclination from Eq. 14

2.2.1 Example with J0437-4715

Parameter	Value
d_s	0.12kpc
Ω_s	144°
V_{ISM}	-50 km s^{-1}

Table 3: Physical Parameters for the Second Screen

We continue our previous example for this pulsar by adding a second screen with parameters in Table 3.

The existence of a unique combination of Ω_o and i_p is shown in Fig. 3.

3 Eccentric Orbits

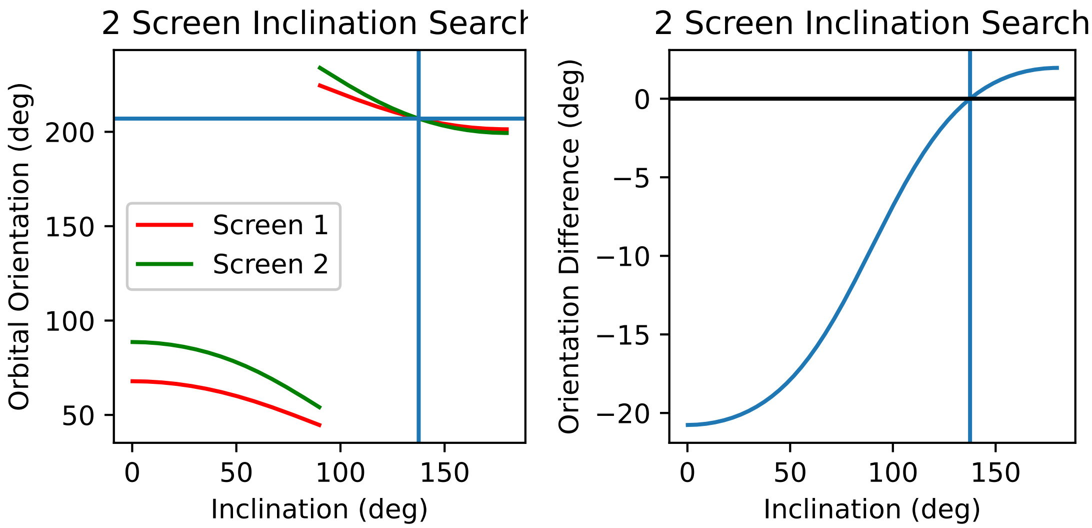
3.1 Earth

Adapting the approach to use the full information about the Earth's orbit can be done quite easily using astropy to calculate the projected velocity of the Earth on the sky in the direction of the pulsar. The dependence of v_{eff} on the α and δ components (where the information is stored) will still be based only on Ω_s and d_{eff}, and so we still measure these parameters and nothing else from the annual variation.

3.2 Pulsar

For the eccentric pulsar orbit, we can write the position in the orbital plane from the true anomaly as

$$\vec{r} = \frac{a_0}{\sin(i_p)}\frac{1-e^2}{1+e\cos(\theta_p)}\left(\cos(\theta_p), \sin(\theta_p)\right) \tag{19}$$



(a) Relationship between i_p and Ω_p

(b) Relationship between $\sin(i_p)$ and d_p

Figure 3: Two screen parameter relations

expanding in terms of e and taking the time derivative gives

$$\vec{v} = \frac{a_0}{\sin(i_p)}(1-e^2)\dot{\theta}_p \left(-\sin(\theta_p) + e\sin(2\theta_p), \cos(\theta_p) - e\cos(2\theta_p)\right) \tag{20}$$

for simplicity, we define the functions

$$\begin{aligned} f_1(\theta_p) &= \sin(\theta_p) - e\sin(2\theta_p) \\ f_2(\theta_p) &= \cos(\theta_p) - e\cos(2\theta_p) \end{aligned} \tag{21}$$

and set $K_p = a_0(1-e^2)\dot{\theta_p}$. Rotating by the longitude of periastron ω and projecting onto the sky now gives

$$\vec{v}_2 = \frac{K_p}{\sin(i_p)} \left(-f_1(\theta_p)\cos(\omega) - f_2(\theta_p)\sin(\omega), (f_2(\theta_p)\cos(\omega) - f_1(\theta_p)\sin(\omega))\cos(i_p)\right) \tag{22}$$

projecting onto the screen gives

$$\vec{v_2} \cdot \vec{s} = A_1 f_1(\theta_p) + A_2 f_2(\theta_p) \tag{23}$$

where

$$\begin{aligned} A_1 &= -\frac{K_p}{\sin(i_p)}(\cos(\omega)\cos(\Delta\Omega_p) + \sin(\omega)\cos(i_p)\sin(\Delta\Omega_p)) \\ A_2 &= \frac{K_p}{\sin(i_p)}(-\sin(\omega)\cos(\Delta\Omega_p) + \cos(\omega)\cos(i_p)\sin(\Delta\Omega_p)) \end{aligned} \tag{24}$$

Since v_{eff} depends on $\frac{d_{eff}}{d_p}\vec{v_2} \cdot \vec{s}$, we are again left with two observables from which to extract three unknown parameters (Ω_p, i_p, and d_p). As for the circular case, a second screen can resolve the degeneracy or information about $\sin(i)$ or d_p can restrict us to only two possible mirrored solutions.